Int. J. Solids Structures, 1973, Vol. 9, pp. 1421 to 1430. Pergamon Press. Printed in Great Britain

A THEOREM FOR THE PLASTIC DESIGN OF REGULAR TWISTLESS GRIDS UNDER CONTINUOUS TRANSVERSE LOADING

M. GRIGORIAN and S. YAGHMAI

Department of Structural Engineering, Arya-Mehr University of Technology, Tehran, Iran

Abstract—A theorem estimating the collapse load of twistless orthotropic grillages of regular formation with any combination of boundary support conditions along the sides of a parallelogram and carrying a continuous distribution of normal nodal loading over the entire surface of the grillage is presented. The validity and the uniqueness of the proposed theorem are established by the upper and the lower bound theorems of the plastic methods of structural analysis in conjunction with the techniques of the finite difference calculus. It is shown by use of this theorem that the previously difficult grillage problems can be solved in a very simple and efficient manner. As practical examples of the applications of the theorem, solutions to the collapse of regular grids under uniform transverse pressure with all possible combinations of boundary support conditions have been presented in tabular form.

NOTATION

X, Y	main coordinates (directions)
x, y	integers defining coordinates of a joint
m, n	integers defining number of bays in X and Y directions respectively
i, j	integers defining coordinates of plastic hinge lines
$M_x, \bar{M}_y$	plastic moment capacity of X and Y beams respectively
$M, \bar{M}$	maximum value of the plastic moment capacity of X and Y beams respectively
$M(x, y), \bar{M}(x, y)$	ultimate bending moment distributions for X and Y beams respectively
$\mu = \bar{M}/M$	coefficient of orthotropy
$p(x, y)$	external transverse nodal loading
$P(x, y)$	ultimate external transverse nodal loading
P	maximum value of the ultimate external nodal load intensity
R	boundary support reaction
$W(x, y), \bar{W}(x, y)$	ultimate load intensity distributions for X and Y beams respectively
$\alpha(y), \beta(x)$	arbitrary continuous functions of y and x respectively
γ	arbitrary constant defining twist
E_x, E_y	forward shift operators
E_x^{-1}, E_y^{-1}	backward shift operators
$\Delta_x = E_x - 1$	forward difference operator
$\nabla_y = 1 - E_y^{-1}$	backward difference operator
$\boxtimes_x = E_x + E_x^{-1} - 2$	second central difference operator

The remainder of the symbols are defined as they first appear in the paper

INTRODUCTION

REGULAR gridwords composed of two sets of parallel beams intersecting at constant angles and supported along the sides of a parallelogram are frequently incorporated in the construction of engineering structures. In spite of their popular use and the abundance of technical literature dealing with the elastic and plastic analysis of grillages and similar structures, no generalized theorem estimating the ultimate load carrying capacity of such systems has yet been developed.

The pioneering study in this subject is due to Heyman [1, 2] who some twenty years ago demonstrated general methods leading to upper and lower bounds to the collapse of transversely loaded grids, concluding that for practical grillages the omission of the torsional resistance of the members has little or no effect on the final load carrying capacity of the structure. Lower bound solutions for a number of grillages worked out by Heyman and by Hodge [3] reveal that the conventional methods of analysis are tedious and often far too involved for everyday use in design. The step-by-step procedures become even more difficult with increasing number of members; this is particularly true for grids with unequal mesh and moments of resistance in the two directions, and more so with different boundary conditions along the four sides.

Since Heyman many methods of analysis for the plastic design of grillages have been proposed, but none seem to present suitable formulae for practical design purposes, i.e. a closed form of generalized solution containing the relevant design variables, namely, the mesh size, the coefficient of orthotropy and the number of beams in the two directions. Recently few such solutions for a number of practical grillages with certain combinations of boundary conditions were produced by the first author [4–6], where use was made of the techniques of the finite difference calculus to formulate and solve the governing equilibrium equation of regular twistless grids. These formulae, although of some practical importance are each limited in application to only one type of loading and boundary condition.

The object of the present work is to introduce a simple generalized theorem for the collapse load of twistless orthotropic grillages of regular formation arbitrarily supported along the sides of a parallelogram and subjected to a continuous distribution of normal nodal forces over the entire surface of the grillage. The theorem eliminates the need for detailed analysis of the grillages and allows the required results to be written down by referring to beam collapse formulae only.

In the forthcoming sections after stating the concept of the theorem two otherwise difficult problems are solved to demonstrate the simplicity of the proposed method. The examples are followed by a lower-bound proof of the theorem which is then confirmed by the upper-bound theorem to demonstrate that the theorem also results in valid unique solutions for the collapse load of the grillage types under consideration. For the purpose of analysis it has been assumed that the beams of the grillage are interconnected by means of vertical shear connectors rather than having rigid connections at the joints. Further, instead of a generalized parallelogram, a rectangular boundary shape has been chosen for the reference grid as in Fig. 1. However, since no flexural–torsional interaction takes place between the intersecting sets, these simplifying measures incur neither loss of generality nor errors in the final results.

THEOREM

If for any twistless orthotropic grillage of regular construction with boundaries forming a parallelogram the variation of the ultimate bending moment capacity of the beams of any of the two sets is the same as the variation of the representative intensity of the nodal loading applied to the same beams, then the ultimate load intensity of the grillage will be given by the sum of the ultimate load intensities of two intersecting beams.

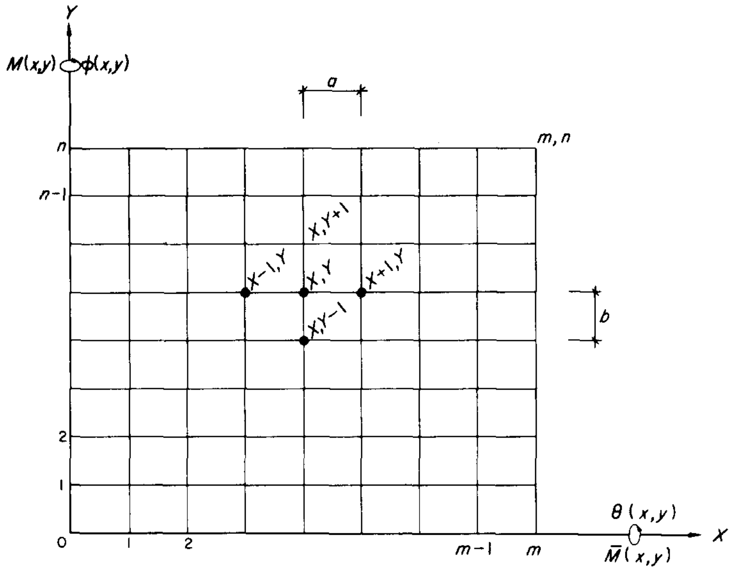


FIG. 1. Reference regular grid, layout and coordinates.

The representative intensity is best defined as the maximum value of the load distribution function, e.g. the apex value in a triangular distribution of loading over the entire span of a beam. In other words if the ultimate load intensity distributions for X and Y beams are represented by

$$W(x, y) = \alpha(y)W(x), \tag{1}$$

and

$$\overline{W}(x, y) = \beta(x)\overline{W}(y) \tag{2}$$

respectively, then the variations of the representative intensities of these load distributions are $\alpha(y)W$ and $\beta(x)\overline{W}$ respectively; and if the plastic moment capacities of the X and Y beams vary as

$$M_x = \alpha(y)M(x), \tag{3}$$

and

$$M_y = \beta(x)\overline{M}(y) \tag{4}$$

respectively then the ultimate load carrying capacity of the entire grillage is given by

$$P(x, y) = W(x, y) + \overline{W}(x, y) \tag{5}$$

where, $\alpha(y) \geq 0$ for all y and $\beta(x) \geq 0$ for all x.

Illustrative example 1

To illustrate the concept of this theorem consider the collapse load of a rectangular orthotropic grillage of regular formation carrying a uniform concentration of normal nodal forces P, with two opposite edges at $x = 0$ and $x = m$ simply supported, and the other two along $y = 0$ and $y = n$ fully fixed. Now turning to the reference diagram in Fig. 1, it may easily be shown that for an independent X beam to collapse under uniform load $W(x, y) = W$, i.e. $\alpha(y) = 1$, the representative collapse intensity is

$$W = \frac{8M}{a(m^2+\delta_m)} \quad \text{where } \delta_m = \{(-1)^m - 1\}/2,$$

and that for an independent Y beam to collapse under uniform load $\overline{W}(x, y) = \overline{W}$, i.e. $\beta(x) = 1$, the representative collapse intensity is

$$\overline{W} = \frac{16\overline{M}}{b(n^2+\delta_n)} \quad \text{where } \delta_n = \{(-1)^n - 1\}/2.$$

(δ_m and δ_n have been introduced to ensure correct answers for beams containing both even and odd number of loaded points.)

Now by equation (5) the collapse load of the whole grillage is

$$P = W + \overline{W} = 8M\left\{\frac{1}{a(m^2+\delta_m)} + \frac{2\mu}{b(n^2+\delta_n)}\right\},$$

a result previously obtained by rigorous analysis [5].

Illustrative example 2

As a second example consider the case of a hydrostatically loaded regular grillage composed of simply-supported horizontal X beams with plastic moment capacities linearly varying with location from zero along $y = n$, (where the pressure is nil) to a maximum value of M at $y = 0$, (where the pressure attains its largest value P) and identical cantilevered beams in the Y direction, with fixed edges along $y = 0$.

Now since $\alpha(y) = (1 - y/n)$ and $\beta(x) = 1$ the representative collapse intensities of independent X and Y beams will be

$$W = \frac{8M}{a(m^2+\delta_m)} \quad \text{and} \quad \overline{W} = \frac{6\mu M}{b(n^2-1)}$$

respectively, and by equation (5), if the beams acted all together the collapse load of the grillage would be

$$P = W + \overline{W} = M\left\{\frac{8}{a(m^2+\delta_m)} + \frac{6\mu}{b(n^2-1)}\right\}.$$

This implies that the distribution of the ultimate external loading is given by

$$P(x, y) = M\left\{\frac{8}{a(m^2+\delta_m)} + \frac{6\mu}{b(n^2-1)}\right\}(1 - y/n);$$

once again a proven result previously obtained by longhand analysis [7].

PROOF

(a) *The lower-bound approach*

As far as the requirement of plastic collapse theorems are concerned, a unique solution for a grid is one which contains statically admissible bending moment fields, which, while remaining compatible with the boundary support conditions, and satisfying the prescribed yield criteria, in turn contains a sufficient number of suitably located maxima or plastic hinges to transform the structure into a mechanism.

Therefore, looking at the problem from a unique solution point of view and studying the governing difference equilibrium equation of the twistless grids,

$$(1/a)\boxbslash_x M(x, y)+(1/b)\boxbslash_y \overline{M}(x, y) = -P(x, y), \tag{6}$$

it may be seen that due to the absence of torsional resistance, equation (6) may be considered to be composed of two parts, each describing the equilibrium state of one set of beams at collapse, i.e.

$$(1/\mathrm{a})\boxbslash_x M(x, y) = -P(x, y)+S(x, y), \tag{7}$$

$$(1/b)\boxbslash_y \overline{M}(x, y) = -S(x, y) \tag{8}$$

where $S(x, y)$ is the unknown shear force acting between the nodes of the two sets. The breaking of equation (6) in this manner may be interpreted as describing the structural action of the grillage as a set of parallel loaded beams, say the beams running in the X direction, supported on top of the beams of the Y direction. However, if the stated conditions of the unique solution are to be satisfied with the individual beams of the grillage collapsing through the same direction as the applied loading then the permissible upper limits of the intensities of the loads of the beams of the two sets become apparent.

$$W(x, y) = P(x, y)-S(x, y), \tag{9}$$

$$\overline{W}(x, y) = S(x, y). \tag{10}$$

Elimination of $S(x, y)$ between equations (9) and (10) suggests the basic form of the theorem, i.e.

$$P(x, y) = W(x, y)+\overline{W}(x, y).$$

Now treating $W(x, y)$ and $\overline{W}(x, y)$ as the ultimate intensities of the loads of the beams of the two sets and referring to the beam equilibrium equations (7) and (8), it may be seen that these second order difference equations will have solutions describable by continuous, bounded functions containing two constants of integration and a parameter defining the magnitude of the loading at collapse, i.e. a sufficient number of constants to satisfy the boundary force, as well as yielding conditions along the spans meanwhile describing a state of collapse through the correct beam mechanisms. Table 1 presents such solutions for constant section, uniformly loaded single span beams, as component parts of the grillage system, with all possible combinations of the boundary conditions. Therefore, it is evident that solutions of the forms presented in Table 1, each, by virtue of the requirements of unique solutions, constitute valid generalized unique solutions for the collapse of the beams under consideration. However, since the beams of the same set have the same boundary conditions and, that in accordance with conditions (1–4), similar distributions of bending moments under similar type of loading can be assigned to each one of them, then they will all collapse through similar mechanisms with their plastic hinges falling along one or more straight lines as the case may require.

Now if the beams of the two sets are each assigned a bending moment distribution of the form described above then their plastic hinge lines will divide the grillage into segments of parallelograms each capable of undergoing rigid body displacements, thus forming a suitable plastic hinge pattern to transform the grillage into a mechanism. This may easily be visualized considering the fact that due to the absence of the twisting restraints the beams of the collapsing segments will fall along the lines of the constant slopes of hyperbolic-paraboloids with undisplaced edges along the supports and a high point at the intersection of two hinge lines or a hinge line and a free boundary; for instance, see

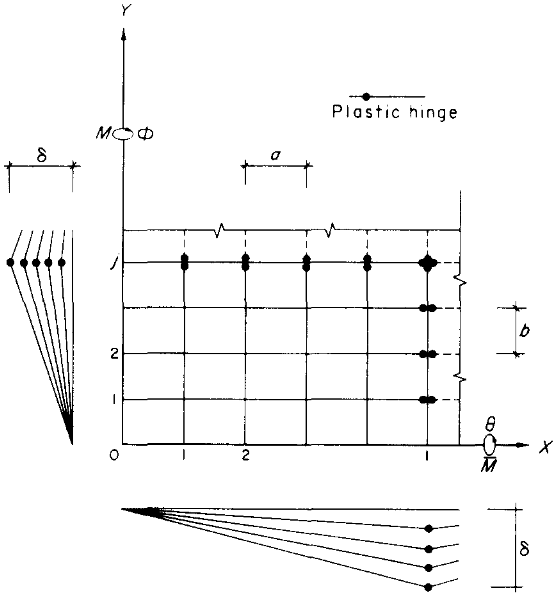
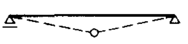
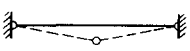
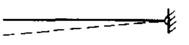


FIG. 2. Rigid body displacements and rotations at collapse.

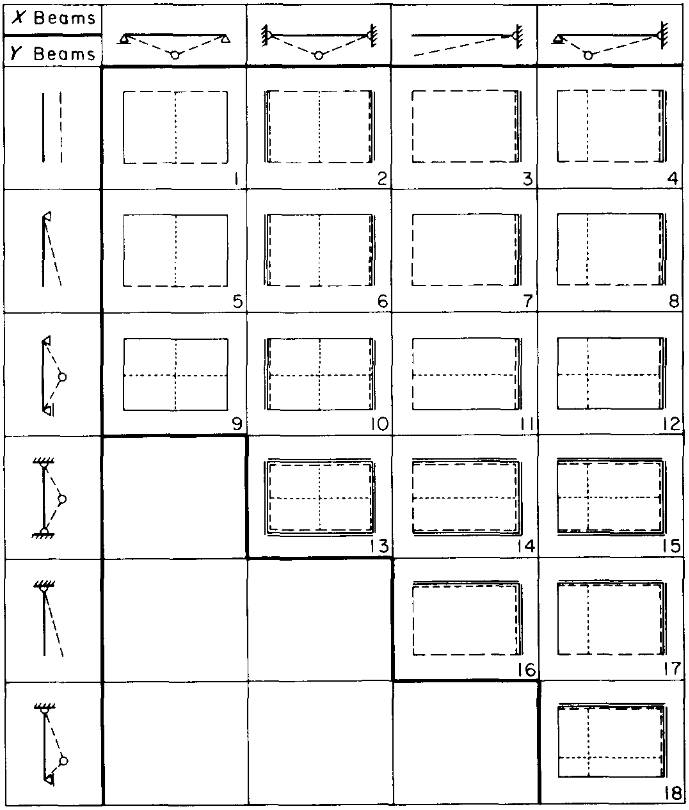
TABLE 1. UNIQUE SOLUTIONS TO THE COLLAPSE OF SINGLE SPAN BEAMS OF CONSTANT CROSS SECTION UNDER UNIFORM DISTRIBUTION OF CONCENTRATED NORMAL LOADS

$\delta_m = [(-1)^m - 1]/2$

Beam, boundary conditions and failure mode	Moment distribution at collapse $M(x)/M =$	Load intensity at collapse $W(x)a/M =$	Vertical reaction at left support $aR_L/M =$	Vertical reaction at right support $aR_R/M =$
 1	$\frac{4(mx-x^2)\dagger}{m^2+\delta_m}$	$\frac{8}{m^2+\delta_m}$	$\frac{4(m-1)}{m^2+\delta_m}$	$\frac{4(m-1)}{m^2+\delta_m}$
 2	$\frac{8(mx-x^2)}{m^2+\delta_m}-1$	$\frac{16}{m^2+\delta_m}$	$\frac{8(m-1)}{m^2+\delta_m}$	$\frac{8(m-1)}{m^2+\delta_m}$
 3	$\frac{x^2+x}{m^2+m}$	$\frac{2}{m^2+m}$	0	$\frac{2m}{m^2+m}$
 4	$\frac{(m^2+j^2)x-(m+j)x^2}{mj(m-j)}$	$\frac{2(m+j)}{mj(m-j)}$	$\frac{(m^2+j^2)-(m+j)}{mj(m-j)}$	$\frac{(m+j)(2m-1)-(m^2+j^2)}{mj(m-j)}$

* j = nearest integer to the real value $(\sqrt{2}-1)m$.
† $m \geq 1$ in all formulae.

TABLE 2. PLASTIC COLLAPSE PATTERNS FOR UNIFORMLY LOADED REGULAR GRIDS WITH ALL COMBINATIONS OF BOUNDARY SUPPORT CONDITIONS ALONG THE SIDES OF A PARALLELOGRAM



Simply-supported edge
Fully-fixed edge
Plastic hinges along fixed edge
Free edge
Plastic hinge line

Fig. 2. To illustrate this point the hinge line patterns of uniformly loaded grids with all possible combinations of boundary support conditions are presented in Table 2.

Therefore, since the grillage equilibrium equation is solved in such a manner as to satisfy the requirements of a lower bound solution meanwhile allowing the grillage to collapse through suitable mechanisms, then the theorem is true and results in unique solution for the collapse load of the types of grids and loadings considered in this paper.

(b) *The upper-bound approach*

As referred to in the preceding section, for a grillage (under the type of loading discussed in this paper) to fail, each collapsing segment being a parallelogram will undergo rigid body displacements with constant twist without stress defined by

$$\gamma = \Delta_x \Delta_y Z(x, y) \tag{11}$$

where $Z(x, y)$ is the displacement function of a collapsing segment. Upon integration equation (11) yields, as expected, a hyperbolic-paraboloidal surface of the form

$$Z(x, y) = \gamma xy. \tag{12}$$

Referring to Fig. 2 it will be seen that to the rigid body displacement $Z(x, y)$ correspond rigid body rotations $\phi(x, y)$ and $\theta(x, y)$ about Y and X axes respectively, where

$$\phi(x, y) = (1/a)\Delta_x Z(x, y) = \gamma y, \tag{13}$$

$$\theta(x, y) = (1/b)\Delta_y Z(x, y) = \gamma x. \tag{14}$$

Therefore, studying the work equation for the loaded grid, say for the segment shown in Fig. 2 it gives,

$$rM \sum_{y=1}^{j} \phi(x, y) + k\bar{M} \sum_{x=1}^{i} \theta(x, y) = \sum_{x=1}^{i} \sum_{y=1}^{j} P(x, y)Z(x, y), \tag{15}$$

or

$$rM\gamma \sum_{y=1}^{j} y + k\bar{M}\gamma \sum_{x=1}^{i} x = \gamma \sum_{x=1}^{i} \sum_{y=1}^{j} P(x, y)xy \tag{16}$$

where ($r = 1$ or 2) and ($k = 1$ or 2) are auxiliary terms generalizing the number of hinge rotations per rigid segment of each beam. Now considering the independent failure of the constituent beams of the same segment and studying the corresponding work equations based on the sum of the internal and external works of each beam under the appropriate collapse intensities it gives for the beams running in the X direction

$$rM\gamma \sum_{y=1}^{j} y = \gamma \sum_{y=1}^{j} \sum_{x=1}^{i} W(x, y)xy, \tag{17}$$

and for the beams running in the Y direction

$$kM\gamma \sum_{x=1}^{i} x = \gamma \sum_{x=1}^{i} \sum_{y=1}^{j} \bar{W}(x, y)xy. \tag{18}$$

Adding equation (17) to equation (18) to obtain the work equation for the whole segment it yields

$$rM\gamma \sum_{y=1}^{j} y + k\bar{M}\gamma \sum_{x=1}^{i} x = \gamma \sum_{x=1}^{i} \sum_{y=1}^{j} \{W(x, y) + \bar{W}(x, y)\}xy. \tag{19}$$

Now comparing the right-hand sides of work equations (16) and (19), it becomes apparent that

$$P(x, y) = W(x, y) + \bar{W}(x, y).$$

This result obtained for one collapsing segment can easily be extended over the entire grillage to subsequently prove the statement of the theorem from an upper-bound point of view.

APPLICATIONS

To demonstrate the applications of the proposed theorem consider the practically important cases of rectangular regular grids subjected to a uniform concentration of normal nodal forces P with arbitrary combination of boundary support conditions as shown in Table 2. The unique collapse load solutions for the beams of all possible cases shown in Table 2 are condensed in Table 1. For instance considering the yet untreated case of such a grillage with three sides fully fixed and the other simply supported, i.e. case 15 from Table 2, the corresponding complete solution immediately follows from the second and fourth lines of Table 1. Thus, since $P = \overline{W} + W$, then from line 2, Table 1,

$$\overline{W} = \frac{16\overline{M}}{(n^2+\delta_n)b},$$

and from line 4, Table 1,

$$W = \frac{2M(m+j)}{mj(m-j)a}.$$

Therefore,

$$P = M\left\{\frac{2(m+j)}{mj(m-j)a} + \frac{16\mu}{(n^2+\delta_n)b}\right\}$$

with the corresponding admissible moment fields

$$M(x) = M\left\{\frac{(m^2+j^2)x-(m+j)x^2}{mj(m-j)}\right\},$$

and

$$M(y) = \mu M\left\{\frac{8(ny-y^2)}{n^2+\delta_n} - 1\right\}.$$

Now since the moment fields describe the exact distribution of forces at collapse they may also be used to determine the support reactions along the boundaries; thus

$$R_{x=0} = \frac{M\{(m^2+j^2)-(m+j)\}}{mj(m-j)a},$$

$$R_{x=m} = \frac{M\{(m+j)(2m-1)-(m^2+j^2)\}}{mj(m-j)a},$$

$$R_{y=0} = R_{y=n} = \frac{8\mu M(n-1)}{(n^2+\delta_n)b},$$

which are directly obtainable from Table 1. As a statical check it might be confirmed that the sum of the reactions along the supports balance the total external load applied to the internal nodes of the grillage, i.e.

$$(m-1)(R_{y=0}+R_{y=n})+(n-1)(R_{x=0}+R_{x=m}) = (m-1)(n-1)P.$$

CONCLUDING REMARKS

A new theorem for the plastic design of certain classes of regular gridworks has been presented. It has been shown by use of this theorem that the previously difficult and extremely tedious grillage problems can now be solved in a very simple and efficient manner.

The theorem presented in this paper is restricted in application to twistless regular grids supported along the sides of a parallelogram with members parallel with the boundaries; further, the loading should be describable by continuous functions of the main variables. Therefore, as illustrated, the theorem is particularly useful for solving grillage problems with uniform, hydrostatic or trapezoidally varying loading over the entire surface of the structure.

As it is known, whenever lower-bound solutions are available they can be used as great aids for the minimum weight design of the structure. The solutions obtained by the present theorem result in unique collapse loads for the grillages considered here and may be used to optimize the total weight of the material used in the construction of these grids. The distribution functions indicate that the normal reactions along the supports are uniform, thus allowing for the more economical design of the edge beams.

The theorem may be extended to study the plastic collapse of slabs of torsionally weak materials as well as orthogonally ribbed plates of negligible torsional resistance, and possibly to investigate the ultimate load behaviour of certain interconnected truss systems.

REFERENCES

[1] J. HEYMAN, The limit design of a transversly loaded square grid. *J. appl. Mech.* **19**, 153 (1952).
[2] J. HEYMAN, The plastic design of grillages. *Engineering* **176**, 804 (1953).
[3] P. G. HODGE, *Plastic Analysis of Structures.* McGraw-Hill (1959).
[4] M. GRIGORIAN, A lower bound solution to the collapse of uniform rectangular grids on simple supports. *Int. J. Mech. Sci.* **13**, 755 (1971).
[5] M. GRIGORIAN, The plastic design of orthotropic grids with fixed supports. *Int. J. Mech. Sci.* **14**, 197 (1972).
[6] M. GRIGORIAN, Plastic design of regular orthotropic grids with two adjacent edges fixed, free or hinged. *J. Strain Analysis* **7**, 279 (1972).
[7] M. GRIGORIAN and V. SCHRICKER, Plastic design of hydrostatically loaded twistless orthotropic grids, *Der Stahlbau* (submitted for publication).

(*Received* 24 *October* 1972; *revised* 20 *February* 1973)

Абстракт—Дается теория оценки нагрузки разрушения для ортотропных ростверков, регулярного очертания, без учета кручения. Граничные условия опирания вдоль боков параллелограмма суть произвольной комбинации. Ростверки подверженны действию сплошной, нормальной нагрузки, по целой поверхности. Путем применения теорем для верхнего и нижнего пределов, на основе методов пластичности расчета конструкций, вместе с методом конечного элемента, определяются важность и единственность предложенной теоремы. Указано, что используя этую теорему можно подсчитать, заранее сложные задачи ростверков, очень простым и полезным способом. В капестве практических примеров применений этих теорем, приводятся решения, в табличной форме, для расчета нагрузки разрушения регулярных ростверков, под влиянием постоянного, поперечного давления, для всех возможных комбинаций граничных условий.